

CURVES OVER HIGHER LOCAL FIELDS

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ABSTRACT. In this work, we prove the vanishing of the two cohomology group of the higher local field. This generalizes the well known propriety for finite field and one dimensional local field. We apply this result to study the arithmetic of curve defined over higher local field.

1. INTRODUCTION

Let k be a finite field or a local field with finite residue field. A well-known fact is the vanishing of the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ for such field. Using the definition of n -dimensional local field (n -local) introduced by Kato, the finite field is seen as a 0-local and the usually local field as an 1-local. A natural question arises in this context is: For an n -local, the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ can be vanished?

Based on class field theory of such fields studied by Kato, we prove the following result:

Theorem 1.1. (*Theorem 3.1*)

If k is a n -local field of characteristic zero, then the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ vanishes.

We apply this result to investigate the class field theory of curves over n -local (section 3). The case $n = 1$, is already obtained by Saito. Let k_1 be an 1-local and X be a proper smooth geometrically irreducible curve over k_1 . To study the fundamental group $\pi_1^{ab}(X)$, Saito in [11], introduced the groups $SK_1(X)$ and $V(X)$ and constructed the maps $\sigma : SK_1(X) \longrightarrow \pi_1^{ab}(X)$ and $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$ where $\pi_1^{ab}(X)^{géo}$ is defined by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{géo} \longrightarrow \pi_1^{ab}(X) \longrightarrow Gal(k_1^{ab}/k_1) \longrightarrow 0$$

The results obtained by Saito in [11] generalized the previous work of Bloch where he is reduced to the good reduction case [11, *Introduction*]. The method of Saito depends on class field theory for two-dimensional local ring having finite residue field [10]. He shows his results for general curve except for the p -primary part in $\text{char } k = p > 0$ case [11, *Section II - 4*]. The remaining p -primary part had been proved by Yoshida in [14].

There is another direction for proving these results pointed out by Douai in [5]. It consists to consider for all l prime to the residual characteristic, the group $\text{Coker } \sigma$ as the dual of the group W_0 of the monodromy weight filtration of $H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$

$$H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0$$

where $\overline{X} = X \otimes_{k_1} \overline{k_1}$ and $\overline{k_1}$ is an algebraic closure of k_1 . This allow him to extend the precedent results to projective smooth surfaces [5].

The aim of this paper is to use a combination of this approach and the theory of the monodromy-weight filtration of degenerating abelian varieties on local fields explained by Yoshida in his paper [14], to study curves over n -local fields.

Let X be a projective smooth curve defined over an n -local field k .

1991 *Mathematics Subject Classification.* 11G25, 14H25.

A well known problem which arises in this context is the structure of the group $\pi_1^{ab}(X)$. But, by class field theory of the n -local field k , it suffices to investigate the group $\pi_1^{ab}(X)^{géo}$ defined by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{géo} \longrightarrow \pi_1^{ab}(X) \longrightarrow Gal(k^{ab}/k) \longrightarrow 0$$

To determinate the group $\pi_1^{ab}(X)^{géo}$, we firstly use Theorem 3.1 to prove the isomorphism:

Proposition 1.2. (*Proposition 4.4*)

$$\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k}$$

Then, by the Grothendick weight filtration on the group $\pi_1^{ab}(\overline{X})_{G_k}$ and assuming the semi-stable reduction, we obtain the structure of the group $\pi_1^{ab}(X)^{géo}$

Theorem 1.3. (*Theorem 4.5*)

The group $\pi_1^{ab}(X)^{géo} \otimes \mathbb{Q}_l$ is isomorphic to $\widehat{\mathbb{Q}_l^r}$ where r is the k -rank of X .

A finite etale covering $Z \rightarrow X$ of X is called a c.s covering, if for any closed point x of X , $x \times_X Z$ is isomorphic to a finite sum of x . We denote by $\pi_1^{c.s}(X)$ the quotient group of $\pi_1^{ab}(X)$ which classifies abelian c.s coverings of X .

To study the class field theory of the curve X , we use the generalized reciprocity map

$$\sigma : SK_n(X) \longrightarrow \pi_1^{ab}(X)$$

where $SK_n(X) = \text{Coker} \left\{ K_{n+1}(K) \xrightarrow{\oplus \partial_v} \bigoplus_{v \in P} K_n(k(v)) \right\}$ and $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$. The group $V(X)$ is defined to be the kernel of the norm map $N : SK_n(X) \longrightarrow K_n(k)$ induced by the norm map $N_{k(v)/k^x} : K_n(k(v)) \longrightarrow K_n(k)$ for all v . This definition is suggested by Saito [11].

The cokernel of σ is the quotient group of $\pi_1^{ab}(X)$ that classifies completely split coverings of X ; that is $\pi_1^{c.s}(X)$. In this context, we obtain the following result:

Proposition 1.4. (*Proposition 4.7*)

The group $\pi_1^{c.s}(X) \otimes \mathbb{Q}_l$ is isomorphic to \mathbb{Q}_l^r , where r is the k -rank of the curve X .

Our paper is organized as follows. Section 2 is devoted to some notations. Section 3 contains the proprieties which we need concerning n -dimensional local field: duality and the vanishing of the second cohomology group. In section 4, we prove a duality theorem for the curve X which allow us to construct the generalized reciprocity map. Finally, in this section, we investigate the groups $\pi_1^{ab}(X)^{géo}$ and $\pi_1^{c.s}(X)$.

For an abelian group M , and a positive integer $n \geq 1$, M/n denotes the group M/nM .

For a scheme Z , and a sheaf \mathcal{F} over the étale site of Z , $H^i(Z, \mathcal{F})$ denotes the i -th étale cohomology group. The group $H^1(Z, \mathbb{Z}/\ell)$ is identified with the group of all continuous homomorphisms $\pi_1^{ab}(Z) \rightarrow \mathbb{Z}/\ell$. If ℓ is invertible on $\mathbb{Z}/\ell(1)$ denotes the sheaf of ℓ -th root of unity and for any integer i , we denote $\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i}$.

For a field L , $K_i(L)$ is the i -th Milnor group. It coincides with the i -th Quillen group for $i \leq 2$. For ℓ prime to $\text{char } L$, there is a Galois symbol

$$h_{\ell, L}^i : K_i L / \ell \rightarrow H^i(L, \mathbb{Z}/\ell(i))$$

which is an isomorphism for $i = 0, 1, 2$ ($i = 2$ is Merkur'jev-Suslin).

3. ON n -DIMENSIONAL LOCAL FIELD

A local field k is said to be n -dimensional *local* (n -local) if there exists the following sequence of fields k_i ($1 \leq i \leq n$) such that

- (i) each k_i is a complete discrete valuation field having k_{i-1} as the residue field of the valuation ring O_{k_i} of k_i , and
- (ii) k_0 is a finite field.

For such a field, and for ℓ prime to $\text{Char}(k)$, the well-known isomorphism

$$(3.1) \quad H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell$$

and for each $i \in \{0, \dots, n+1\}$ a perfect duality

$$(3.2) \quad H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j)) \rightarrow H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell$$

hold.

The class field theory for such fields is summarized as follows: There is a map

$h : K_n(k) \rightarrow \text{Gal}(k^{ab}/k)$ which generalizes the classical reciprocity map for usually local fields. This map induces an isomorphism $K_n(k)/N_{L/k}K_n(L) \simeq \text{Gal}(L/k)$ for each finite abelian extension L of k . Furthermore, the canonical pairing

$$(3.3) \quad H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \times K_n(k) \rightarrow H^{n+1}(k, \mathbb{Q}_l/\mathbb{Z}_l(n)) \simeq \mathbb{Q}_l/\mathbb{Z}_l$$

induces an injective homomorphism

$$(3.4) \quad H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow \text{Hom}(K_n(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

It is well-known that the group $H^2(M, \mathbb{Q}/\mathbb{Z})$ vanishes when M is a finite field or usually local field. Next, we prove the same result for n -local field:

Theorem 3.1. *If k is a n -local field of characteristic zero, then the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ vanishes.*

Proof. We proceed as in the proof of theorem 4 of [12]. It is enough to prove that $H^2(k, \mathbb{Q}_l/\mathbb{Z}_l)$ vanishes for all l and when k contains the group μ_l of l -th roots of unity. For this, we prove that multiplication by l is injective. That is, we have to show that the coboundary map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\delta} H^2(k, \mathbb{Z}/l\mathbb{Z})$$

is injective.

By assumption on k , we have

$$H^2(k, \mathbb{Z}/l\mathbb{Z}) \simeq H^2(k, \mu_l) \simeq \mathbb{Z}/\ell$$

The last isomorphism is well-known for one-dimensional local field and was generalized to non archimedean and locally compact fields by Shatz in [13]. The proof is now reduced to the fact that $\delta \neq 0$.

Now, $\delta(\Phi) = 0$ if and only if Φ is a l -th power, and Φ is a l -th power if and only if Φ is trivial on μ_l . Thus, it is sufficient to construct an homomorphism $K_n(k) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$ which is non trivial on μ_l .

Let i be the maximal natural number such that k contains a primitive l^i -th root of unity. Then, the image ξ of a primitive l^i -th root of unity under the composite map

$$k^x/k^{xl} \simeq H^1(k, \mu_l) \simeq H^1(k, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)$$

is not zero. Thus, the injectivity of the map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \text{Hom}(K_n(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

gives rise to a character which is non trivial on μ_l . □

Remark 3.2. This proof is inspired by the proof of Proposition 7 of Kato [8]

4. CURVES OVER n -LOCAL FIELD

Let k be an n -local field of characteristic zero and X a smooth projective curve defined over k .

We recall that we denote:

$K = K(X)$ its function field,

P : set of closed points of X , and for $v \in P$,

$k(v)$: the residue field at $v \in P$

The residue field of k is denoted by k_{n-1} .

Denote by k_s a separable closure of k and $\overline{X} = X \otimes_k k_s$. Then, we will consider the spectral sequence

$$S(j) \quad H^p(X, H^q(\overline{X}, \mathbb{Z}/\ell(j))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(j))$$

To construct a generalized reciprocity map for X , we need the following duality theorem:

Theorem 4.1. *For all ℓ prime to residual characteristics, the isomorphism*

$$(4.1) \quad H^{3+n}(X, \mathbb{Z}/\ell(1+n)) \simeq \mathbb{Z}/\ell$$

and the perfect pairing

$$(4.2) \quad H^1(X, \mathbb{Z}/\ell) \times H^{2+n}(X, \mathbb{Z}/\ell(1+n)) \longrightarrow H^{3+n}(X, \mathbb{Z}/\ell(1+n)) \simeq \mathbb{Z}/\ell$$

occur. Furthermore, this duality is compatible with duality (3.2) in the sense that the commutative diagram

$$(4.3) \quad \begin{array}{ccccccc} H^1(X, \mathbb{Z}/\ell) & \times & H^{2+n}(X, \mathbb{Z}/\ell(1+n)) & \longrightarrow & H^{3+n}(X, \mathbb{Z}/\ell(1+n)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \\ \downarrow i^* & & \uparrow i_* & & \uparrow i_* & & \parallel \\ H^1(k(v), \mathbb{Z}/\ell) & \times & H^n(k(v), \mathbb{Z}/\ell(n)) & \longrightarrow & H^{n+1}(k(v), \mathbb{Z}/\ell(n)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \end{array}$$

holds, where i^* is the map on H^i induced from the map $v \longrightarrow X$ and i_* is the Gysin map.

Proof. We reformulate the proof of Theorem 2.1 of [6] without the assumption of good reduction done there. We use the spectral sequence ($j = n + 1$)

$$S(1+n) \quad H^p(k, H^q(\overline{X}, \mathbb{Z}/\ell(1+n))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(1+n))$$

As k is n -dimensional local field, we have $H^{n+2}(k, M) = 0$ for any torsion module M , we obtain:

$$\begin{aligned} H^{3+n}(X, \mathbb{Z}/\ell(1+n)) &\subseteq H^{n+1}(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n))) \\ &\subseteq H^{n+1}(k, \mathbb{Z}/\ell(n)) \\ &\subseteq \mathbb{Z}/\ell \quad \text{by (3.1)} \end{aligned}$$

We prove now the duality (4.2). The filtration of the group $H^{2+n}(X, \mathbb{Z}/\ell(1+n))$ is

$$H^{2+n}(X, \mathbb{Z}/\ell(1+n)) = E_n^{2+n} \supseteq E_{n+1}^{2+n} \supseteq 0$$

which leads to the exact sequence

$$0 \rightarrow E_\infty^{n+1,1} \rightarrow H^{2+n}(X, \mathbb{Z}/\ell(1+n)) \rightarrow E_\infty^{n,2} \rightarrow 0$$

Since $E_2^{p,q} = 0$ for all $p \geq n+2$ or $q \geq 3$, we see that

$$E_2^{n,2} = E_3^{n,2} = \dots = E_\infty^{n,2}.$$

The same argument yields

$$E_3^{n+1,1} = E_4^{n+1,2} = \dots = E_\infty^{n+1,2}$$

and $E_3^{n+1,1} = \text{Coker } d_2^{n-1,2}$ where $d_2^{n-1,2}$ is the map

$$H^{n-1}(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n))) \rightarrow H^{n+1}(k, H^1(\overline{X}, \mathbb{Z}/\ell(1+n))).$$

Hence, we obtain the exact sequence

$$(4.4) \quad 0 \rightarrow \text{Coker } d_2^{n-1,2} \rightarrow H^{2+n}(X, \mathbb{Z}/\ell(1+n)) \rightarrow H^n(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n))) \rightarrow 0$$

Combining duality (3.2) for k and Poincaré duality, we deduce that the group $H^0(k, H^1(\overline{X}, \mathbb{Z}/\ell(1+n)))$ is dual to the group $H^{n+1}(k, H^1(\overline{X}, \mathbb{Z}/\ell(1+n)))$ and the group $H^2(k, H^0(\overline{X}, \mathbb{Z}/\ell))$ is dual to the group $H^{n-1}(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n)))$. On the other hand, we have the commutative diagram (4.5)

$$\begin{array}{ccccccc} H^{n-1}(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n))) & \times & H^2(k, H^0(\overline{X}, \mathbb{Z}/\ell)) & \rightarrow & H^{n+1}(k, \mathbb{Z}/\ell(n)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \\ \downarrow \wr & & \uparrow & & \parallel & & \parallel \\ H^{n+1}(k, H^1(\overline{X}, \mathbb{Z}/\ell(1+n))) & \times & H^0(k, H^1(\overline{X}, \mathbb{Z}/\ell)) & \rightarrow & H^{n+1}(k, \mathbb{Z}/\ell(n)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \end{array}$$

given by the cup products and the spectral sequence $S(j)$, using the same argument as ([1], diagram 46). We infer that $\text{Coker } d_2^{n-1,2}$ is the dual of $\text{Ker}' d_2^{0,1}$ where $'d_2^{0,1}$ is the boundary map for the spectral sequence

$$(4.6) \quad 'E_2^{p,q} = H^p(k, H^q(\overline{X}, \mathbb{Z}/\ell)) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell)$$

Similarly, the group $H^n(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n)))$ is dual to the group $H^1(k, H^0(\overline{X}, \mathbb{Z}/\ell(1+n)))$. The required duality is deduced from the following commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Coker } d_2^{n-1,3} & \rightarrow & H^{3+n}(X, \mathbb{Z}/\ell(1+n)) & \rightarrow & H^n(k, H^2(\overline{X}, \mathbb{Z}/\ell(1+n))) & \rightarrow & 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 \rightarrow (\text{Ker}' d_2^{0,1})^\vee & \rightarrow & (H^1(X, \mathbb{Z}/\ell))^\vee & \rightarrow & (H^1(k, H^0(\overline{X}, \mathbb{Z}/\ell(1+n))))^\vee & \rightarrow & 0 \end{array}$$

where the upper exact sequence is (4.4) and the bottom exact sequence is the dual of the well-known exact sequence

$$0 \rightarrow 'E_2^{1,0} \rightarrow H^1(X, \mathbb{Z}/\ell) \rightarrow \text{Ker}' d_2^{0,1} \rightarrow 0$$

deduced from the spectral sequence (4.6) and where $(M)^\vee$ denotes the dual $Hom(M, \mathbb{Z}/\ell)$ for any \mathbb{Z}/ℓ -module M .

Finally, to obtain the last part of the theorem, we remark that the commutativity of the diagram (4.3) is obtained via a same argument (projection formula ([9], VI6.5) and compatibility of traces ([9], VI11.1) as [1] to establish the commutative diagram in the proof of assertion ii) at page 791. \square

Remark 4.2. 1) If $n = 1$, we find the duality theorem obtained by Saito in [14].

2) If $n = 2$, we obtain a duality which is analogue to a duality for scheme associated to two-dimensional local ring [2]. For general n , the analogy is explained in [3].

4.1. The reciprocity map. We introduce the group $SK_n(X)/\ell$:

$$SK_n(X)/\ell = \text{Coker} \left\{ K_{n+1}(K)/\ell \xrightarrow{\oplus \partial_v} \bigoplus_{v \in P} K_n(k(v))/\ell \right\}$$

where $\partial_v : K_{n+1}(K) \longrightarrow K_n(k(v))$ is the boundary map in K-Theory. It will play an important role in class field theory for X as pointed out by Saito in the introduction of [11]. In this section, we construct a map

$$\sigma/\ell : SK_n(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

which describe the class field theory of X .

By considering the Zariskien sheaf $\mathcal{H}^i(\mathbb{Z}/\ell(n+1))$, $i \geq 1$ associated to the presheaf $U \longrightarrow H^i(U, \mathbb{Z}/\ell(n+1))$, it is easy to construct a map $\sigma/\ell : SK_n(X)/\ell \longrightarrow H^{n+2}(X, \mathbb{Z}/\ell(n+1))$.

In fact: By definition of $SK_n(X)/\ell$, we have the exact sequence

$$K_{n+1}(K)/\ell \longrightarrow \bigoplus_{v \in P} K_n(k(v))/\ell \longrightarrow SK_n(X)/\ell \longrightarrow 0$$

On the other hand, it is known that the following diagram is commutative:

$$\begin{array}{ccc} K_{n+1}(K)/\ell & \longrightarrow & \bigoplus_{v \in P} K_n(k(v))/\ell \\ \downarrow h^{n+1} & & \downarrow h^n \\ H^{n+1}(K, \mathbb{Z}/\ell(n+1)) & \longrightarrow & \bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) \end{array}$$

where h^n, h^{n+1} are the Galois symbols. This yields the existence of a morphism

$$h : SK_n(X)/\ell \longrightarrow H^1(X_{Zar}, \mathcal{H}^{n+1}(\mathbb{Z}/\ell(n+1)))$$

taking in account the exact sequence

$$H^{n+1}(K, \mathbb{Z}/\ell(n+1)) \longrightarrow \bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) \longrightarrow H^1(X_{Zar}, \mathcal{H}^{n+1}(\mathbb{Z}/\ell(n+1))) \longrightarrow 0$$

obtained from the spectral sequence

$$H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}/\ell(n+1))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(n+1))$$

This morphism h fit in the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & K_{n+1}(K)/\ell & \longrightarrow & \bigoplus_{v \in P} K_n(k(v))/\ell & \longrightarrow & SK_n(X)/\ell & \longrightarrow 0 \\
& \downarrow h^{n+1} & & \downarrow h^n & & \downarrow h & \\
0 \longrightarrow & H^{n+1}(K, \mathbb{Z}/\ell(n+1)) & \longrightarrow & \bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) & \longrightarrow & H^1(X_{Zar}, \mathcal{H}^{n+1}(\mathbb{Z}/\ell(n+1))) & \longrightarrow 0
\end{array}$$

On the other hand the spectral sequence

$$H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}/\ell(n+1))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(n+1))$$

induces the exact sequence

$$\begin{aligned}
(4.7) \quad 0 \longrightarrow H^1(X_{Zar}, \mathcal{H}^{n+1}(\mathbb{Z}/\ell(n+1))) &\xrightarrow{e} H^{n+2}(X, \mathbb{Z}/\ell(n+1)) \\
&\longrightarrow H^0(X_{Zar}, \mathcal{H}^{n+2}(\mathbb{Z}/\ell(n+1))) \longrightarrow H^2(X_{Zar}, \mathcal{H}^{n+1}(\mathbb{Z}/\ell(n+1))) = 0
\end{aligned}$$

Composing h and e , we get the map

$$SK_n(X)/\ell \longrightarrow H^{n+2}(X, \mathbb{Z}/\ell(n+1)).$$

Finally the group $H^{n+2}(X, \mathbb{Z}/\ell(n+1))$ is identified to the group $\pi_1^{ab}(X)/\ell$ by the duality (4.2)

Hence, we obtain the map

$$\sigma/\ell : SK_n(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

Remark 4.3. The spectral sequence

$$H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}/\ell(n+1))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(n+1))$$

implies that the group $H^0(X_{Zar}, \mathcal{H}^{n+2}(\mathbb{Z}/\ell(n+1)))$ coincides with the kernel of the map

$$H^{n+2}(K, \mathbb{Z}/\ell(n+1)) \longrightarrow \bigoplus_{v \in P} H^{n+1}(k(v), \mathbb{Z}/\ell(n))$$

and by localization in étale cohomology

$$\bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) \longrightarrow H^{n+2}(X, \mathbb{Z}/\ell(n+1)) \longrightarrow H^{n+2}(K, \mathbb{Z}/\ell(n+1)) \longrightarrow \bigoplus_{v \in P} H^{n+1}(k(v), \mathbb{Z}/\ell(n))$$

and taking in account (4.7), we see that $H^1(X_{Zar}, \mathcal{H}^{n+2}(\mathbb{Z}/\ell(n+1)))$ is the image of the Gysin map

$$\bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) \xrightarrow{g} H^{n+2}(X, \mathbb{Z}/\ell(n+1))$$

and consequently the morphism g factorize through $H^1(X_{Zar}, \mathcal{H}^{n+2}(\mathbb{Z}/\ell(n+1)))$

$$\begin{array}{ccc}
\bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) & \xrightarrow{g} & H^{n+2}(X, \mathbb{Z}/\ell(n+1)) \\
\searrow & & \nearrow \\
& H^1(X_{Zar}, \mathcal{H}^{n+2}(\mathbb{Z}/\ell(n+1))) &
\end{array}$$

Then, we deduce the following commutative diagram

$$\begin{array}{ccccc}
K_{n+1}(K)/\ell & \longrightarrow & \bigoplus_{v \in P} K_n(k(v))/\ell & \longrightarrow & SK_n(X)/\ell \longrightarrow 0 \\
\downarrow h^{n+1} & & \downarrow h^n & & \downarrow h \\
H^{n+1}(K, \mathbb{Z}/\ell(n+1)) & \longrightarrow & \bigoplus_{v \in P} H^n(k(v), \mathbb{Z}/\ell(n)) & \longrightarrow & H^1(X_{Zar}, \mathcal{H}^{n+2}(\mathbb{Z}/\ell(n+1))) \longrightarrow 0 \\
& & \downarrow g & \swarrow e & \\
& & \pi_1^{ab}(X)/\ell = H^{n+2}(X, \mathbb{Z}/\ell(n+1)) & &
\end{array}$$

The map h is surjective, if we assume the conjecture 1 of Kato [7, page 608]. Without assuming this conjecture, we see that the cokernel of

$$\sigma/\ell : SK_n(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

contains the cokernel of the Gysin map g which is the dual of the kernel of the map

$$(4.8) \quad H^1(X, \mathbb{Z}/\ell) \longrightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell)$$

4.2. The group $\pi_1^{ab}(X)^{géo}$. In his paper [11], Saito don't prove the p -primary part in the char $k = p > 0$ case. This case was developed by Yoshida in [14]. His method is based on the theory of monodromy-weight filtration of degenerating abelian varieties on local fields. In this work, we use this approach to investigate the group $\pi_1^{ab}(X)^{géo}$. As mentioned by Yoshida in [14, section 2] Grothendieck's theory of monodromy-weight filtration on Tate module of abelian varieties are valid where the residue field is arbitrary perfect field.

We assume the semi-stable reduction and choose a regular model \mathcal{X} of X over $Spec O_k$, by which we mean a two dimensional regular scheme with a proper birational morphism

$f : \mathcal{X} \longrightarrow Spec O_k$ such that $\mathcal{X} \otimes_{O_k} k \simeq X$ and if \mathcal{X}_s designates the special fiber $\mathcal{X} \otimes_{O_k} k_1$, then $Y = (\mathcal{X}_s)_{red}$ is a curve defined over the residue field k_1 such that any irreducible component of Y is regular and it has ordinary double points as singularity.

Let $V(X)$ be the kernel of the norm map $N : SK_n(X) \longrightarrow K_n(k)$ induced by the norm map $N_{k(v)^x/k^x} : K_n(k(v)) \longrightarrow K_n(k)$ for all v . Then, we obtain a map $\tau/\ell : V(X)/\ell \longrightarrow \pi_1^{ab}(X)^{géo}/\ell$ and a commutative diagram

$$\begin{array}{ccccc}
V(X)/\ell & \longrightarrow & SK_n(X)/\ell & \longrightarrow & K_n(k)/\ell \\
\downarrow \tau/\ell & & \downarrow \sigma/\ell & & \downarrow h/\ell \\
\pi_1^{ab}(X)^{géo}/\ell & \longrightarrow & \pi_1^{ab}(X)/\ell & \longrightarrow & Gal(k^{ab}/k)/\ell
\end{array}$$

where the map $h/\ell : K_n(k)/\ell \longrightarrow Gal(k^{ab}/k)/\ell$ is the one obtained by class field theory of k (section 3). From this diagram we see that the group $Coker \tau/\ell$ is isomorphic to the group $Coker \sigma/\ell$. Next, we investigate the map τ/ℓ .

We start by the following result which is a consequence of the structure of the n -local field k :

Proposition 4.4. *There is an isomorphism*

$$\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k}$$

where $\pi_1^{ab}(\overline{X})_{G_k}$ is the group of coinvariants under $G_k = Gal(k^{ab}/k)$.

Proof. As in the proof of Lemma 4.3 of [14], this is an immediate consequence of (Theorem 3.1). □

Now, we are able to deduce the structure of the group $\pi_1^{ab}(X)^{géo}$

Theorem 4.5. *The group $\pi_1^{ab}(X)^{géo} \otimes \mathbb{Q}_l$ is isomorphic to $\widehat{\mathbb{Q}_l}^r$*

where r is the k -rank of X .

Proof. By the preceding proposition, we have the isomorphism $\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k}$. On the other hand the group $\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l$ admits the filtration [14, Lemma 4.1 and section 2]

$$W_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) = \pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l \supseteq W_{-1}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) \supseteq W_{-2}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)$$

But; by assumption; the curve X admits a semi-stable reduction, then the group

$Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) = W_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) / W_{-1}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)$ has the following structure

$$0 \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)_{tor} \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) \longrightarrow \widehat{\mathbb{Q}_l}^{r'} \longrightarrow 0$$

where r' is the k -rank of X . This is confirmed by Yoshida [12, section 2], independently of the finiteness of the residue field of k considered in his paper. The integer r' is equal to the integer $r = H^1(|\overline{\Gamma}|, \mathbb{Q}_l) = H^1(|\Gamma|, \mathbb{Q}_l)$ by assuming that the irreducible components and double points of \overline{Y} are defined over k_{n-1} . \square

4.3. The group $\pi_1^{c.s.}(X)$.

Definition 4.6. Let Z be a Noetherian scheme. A finite etale covering $f : W \rightarrow Z$ is called a c.s covering if for any closed point z of Z , $z \times_Z W$ is isomorphic to a finite scheme-theoretic sum of copies of z . We denote $\pi_1^{c.s.}(Z)$ the quotient group of $\pi_1^{ab}(Z)$ which classifies abelian c.s coverings of Z .

In this context, the group $\pi_1^{c.s.}(X) / \ell$ coincides with the closure of the image of σ / ℓ .

We assume the semi-stable reduction and choose a regular model $f : \mathcal{X} \rightarrow Spec O_k$ of X over $Spec O_k$ as in subsection 4.2.

If \mathcal{X}_s designates the special fiber $\mathcal{X} \otimes_{O_k} k_1$, then $Y = (\mathcal{X}_s)_{red}$ is a curve defined over the residue field k_1 such that any irreducible component of Y is regular and it has ordinary double points as singularity.

Let $\overline{Y} = Y \otimes_{k_{n-1}} \overline{k_{n-1}}$, where $\overline{k_{n-1}}$ is an algebraic closure of k_{n-1} and

$$\overline{Y}^{[p]} = \bigsqcup_{i_j < i_1 < \dots < i_p} \overline{Y}_{i_j} \cap \overline{Y}_{i_1} \cap \dots \cap \overline{Y}_{i_p}, (\overline{Y}_i)_{i \in I} = \text{collection of irreducible components of } \overline{Y}.$$

Let $|\overline{\Gamma}|$ be a realization of the dual graph $\overline{\Gamma}$, then the group $H^1(|\overline{\Gamma}|, \mathbb{Q}_l)$ coincides with the group $W_0(H^1(\overline{Y}, \mathbb{Q}_l))$ constituted of elements of weight 0 for the filtration

$$H^1(\overline{Y}, \mathbb{Q}_l) = W_1 \supseteq W_0 \supseteq 0$$

of $H^1(\overline{Y}, \mathbb{Q}_l)$ deduced from the spectral sequence

$$E_1^{p,q} = H^q(\overline{Y}^{[p]}, \mathbb{Q}_l) \implies H^{p+q}(\overline{Y}, \mathbb{Q}_l)$$

For details see [4] and [5].

Now, if we assume further that the irreducible components and double points of \overline{Y} are defined over k_{n-1} , then the dual graph $\overline{\Gamma}$ of \overline{Y} go down to k_{n-1} and we obtain the injection

$$W_0(H^1(\overline{Y}, \mathbb{Q}_l)) \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l)$$

Proposition 4.7. *The group $\pi_1^{c.s}(X) \otimes \mathbb{Q}_l$ is isomorphic to \mathbb{Q}_l^r , where r is the k -rank of the curve X*

Proof. By (4.8), we see that it suffices to prove that the kernel of the map

$$\alpha : H^1(X, \mathbb{Q}_l) \longrightarrow \prod_{v \in P} H^1(k(v), \mathbb{Q}_l)$$

contains $W_0(H^1(\bar{Y}, \mathbb{Q}_l))$. The group $W_0 = W_0(H^1(\bar{Y}, \mathbb{Q}_l))$ is calculated as the homology of the complex

$$H^0(\bar{Y}^{[0]}, \mathbb{Q}_l) \longrightarrow H^0(\bar{Y}^{[1]}, \mathbb{Q}_l) \longrightarrow 0$$

Hence $W_0 = H^0(\bar{Y}^{[1]}, \mathbb{Q}_l) / \text{Im}\{H^0(\bar{Y}^{[0]}, \mathbb{Q}_l) \longrightarrow H^0(\bar{Y}^{[1]}, \mathbb{Q}_l)\}$. Thus, it suffices to prove the vanishing of the composing map

$$H^0(\bar{Y}^{[1]}, \mathbb{Q}_l) \longrightarrow W_0 \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l) \longrightarrow H^1(k(v), \mathbb{Q}_l)$$

for all $v \in P$.

Let z_v be the 0-cycle in \bar{Y} obtained by specializing v , which induces a map $z_v^{[1]} \longrightarrow \bar{Y}^{[1]}$. Consequently, the map $H^0(\bar{Y}^{[1]}, \mathbb{Q}_l) \longrightarrow H^1(k(v), \mathbb{Q}_l)$ factors as follows

$$\begin{array}{ccc} H^0(\bar{Y}^{[1]}, \mathbb{Q}_l) & \longrightarrow & H^1(k(v), \mathbb{Q}_l) \\ \searrow & & \nearrow \\ & H^0(z_v^{[1]}, \mathbb{Q}_l) & \end{array}$$

But the trace $z_v^{[1]}$ of $\bar{Y}^{[1]}$ on z_v is empty. This implies the vanishing of $H^0(z_v^{[1]}, \mathbb{Q}_l)$. □

Corollary 4.8. *The map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$ has finite image*

Proof. By the diagram in subsection 4.2, the group $\text{Coker } \tau/l$ is isomorphic to the group $\text{Coker } \sigma/l$. Hence, the result is deduced from Theorem 4.5 and the later proposition. □

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